# ECE 443/518 – Computer Cyber Security
# Lecture 08 Euclidean Algorithm, Fermat's Little Theorem

Professor Jia Wang
Department of Electrical and Computer Engineering
Illinois Institute of Technology

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# Outline

Euclidean Algorithm

Fermat's Little Theorem

# Reading Assignment

- This lecture: UC 6.3
- Next lecture: UC 6, 7, except 7.6

# Outline


Euclidean Algorithm

Fermat's Little Theorem

# Euclidean Algorithm

**Input:** two integers $a \geq b > 0$

$r_0 = a,\ r_1 = b,\ i = 1$

**Do**:

  $i = i + 1$

  $r_i = r_{i-2} \bmod r_{i-1}$

**While** $r_i \neq 0$

**Output:** $gcd(a, b) = r_{i-1}$

- Example: $r_0 = 27, r_1 = 21, r_2 = 6, r_3 = 3, r_4 = 0$
  - In practice, there is no need to keep each $r_k$ – we use them just for ease of presentation.
- For a proof of correctness
  - $r_{i-1} = gcd(0, r_{i-1}) = gcd(r_i, r_{i-1}) = \ldots = gcd(r_k, r_{k-1})$
    $= gcd(r_{k-2} \bmod r_{k-1}, r_{k-1}) = gcd(r_{k-2}, r_{k-1}) = \ldots$
    $= gcd(r_1, r_0) = gcd(a, b)$
  - $r_1 > r_2 > \ldots > r_{i-1} > r_i = 0$
- Is this algorithm better than the simple one?

# Time Complexity of Euclidean Algorithm

- Let $q_{k-1} = \lfloor \frac{r_{k-2}}{r_{k-1}} \rfloor$. Since $r_{k-2} \geq r_{k-1}$, $q_{k-1} \geq 1$. So,

$$r_{k-2} = q_{k-1} r_{k-1} + r_k \geq r_{k-1} + r_k \geq 2r_k, \forall k = 2, 3, \ldots, i.$$

- For $i$ being odd, we have,

$$a = r_0 \geq 2r_2 \geq 2^2 r_4 \geq \cdots \geq 2^{\frac{i-1}{2}} r_{i-1} \geq 2^{\frac{i-1}{2}}.$$

  - Similar for $i$ being even.
- The loop iterates $O(\log a) = O(N)$ rounds.
  - Overall the time complexity is $O(N^3)$.
- GCD can be computed efficiently in polynomial time.
  - What is the complexity to obtain any divisor of $a$ that is not 1 or $a$? Or to prove that $a$ is a prime number?

# Extended Euclidean Algorithm (EEA)

| | |
|---|---|
| Input: | two integers $a \geq b > 0$ |
| | $r_0 = a,\ r_1 = b,\ s_0 = 1,\ t_0 = 0,\ s_1 = 0,\ t_1 = 1,\ i = 1$ |
| | **Do**: |
| | $i = i + 1$ |
| | $r_i = r_{i-2} \bmod r_{i-1},\ q_{i-1} = \lfloor \frac{r_{i-2}}{r_{i-1}} \rfloor$ |
| | $s_i = s_{i-2} - q_{i-1}s_{i-1},\ t_i = t_{i-2} - q_{i-1}t_{i-1}$ |
| | **While** $r_i \neq 0$ |
| Output: | $gcd(a, b) = r_{i-1},\ s = s_{i-1},\ t = t_{i-1}$ |

- Same time complexity as Euclidean Algorithm: $O(N^3)$
  - Same rounds of iterations. Additional calculations do not increase complexity.
- Anything special?

# Extended Euclidean Algorithm (EEA Cont.)

| $k$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $r_k$ | 27 | 21 | 6 | 3 | 0 |
| $\begin{pmatrix} 1 \\ -q_{k-1} \end{pmatrix}$ | | | $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ | $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ | $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ |
| $\begin{pmatrix} s_k \\ t_k \end{pmatrix} = \begin{pmatrix} s_{k-2} & s_{k-1} \\ t_{k-2} & t_{k-1} \end{pmatrix} \begin{pmatrix} 1 \\ -q_{k-1} \end{pmatrix}$ | 1<br>0 | 0<br>1 | 1<br>−1 | −3<br>4 | 7<br>−9 |

- Starting with $\begin{pmatrix} r_0 & r_1 \end{pmatrix} = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} s_0 & s_1 \\ t_0 & t_1 \end{pmatrix}$. We have,

$$\begin{aligned} \begin{pmatrix} r_1 & r_2 \end{pmatrix} &= \begin{pmatrix} r_0 & r_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -q_1 \end{pmatrix} = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} s_0 & s_1 \\ t_0 & t_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -q_1 \end{pmatrix} \\ &= \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} s_1 & s_2 \\ t_1 & t_2 \end{pmatrix} \end{aligned}$$

- So we can prove $\begin{pmatrix} r_{i-1} & r_i \end{pmatrix} = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} s_{i-1} & s_i \\ t_{i-1} & t_i \end{pmatrix}$.
- In other words, $as + bt = as_{i-1} + bt_{i-1} = r_{i-1} = gcd(a, b)$

# Solve Modular Algebra Equations

- $ax \equiv b \pmod{m}$
  - Assume $gcd(a, m) = 1$.
  - Apply EEA to find $s$ and $t$ such that $as + mt = 1$.
  - Solution: $x \equiv bs \pmod{m}$
  - Check: $ax \equiv abs \equiv b(1 - mt) \equiv b - bmt \equiv b \pmod{m}$.
- Time complexity is $O(N^3)$, dominated by EEA.

# Examples 1

- Solve $5x \equiv 1 \pmod{192}$.

| $k$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $r_k$ | 192 | 5 | | | |
| $\begin{pmatrix} 1 \\ -q_{k-1} \end{pmatrix}$ | | | | | |
| $\begin{pmatrix} s_k \\ t_k \end{pmatrix}$ | $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ | $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ | | | |

  - $192 \cdot (-2) + 5 \cdot 77 = 1$
  - $x \equiv 77 \pmod{192}$

# Solve System of Modular Algebra Equations

- $x \equiv a_1 \pmod{m_1}$, $x \equiv a_2 \pmod{m_2}$
  - A.k.a. Chinese Remainder Theorem.
  - Assume $gcd(m_1, m_2) = 1$.
  - Apply EEA to find $s$ and $t$ such that $m_1 s + m_2 t = 1$.
  - Solution: $x \equiv a_1 m_2 t + a_2 m_1 s \pmod{m_1 m_2}$
  - Check: $x \equiv a_1 m_2 t \equiv a_1(1 - m_1 s) \equiv a_1 \pmod{m_1}$.
    $x \equiv a_2 m_1 s \equiv a_2(1 - m_2 t) \equiv a_2 \pmod{m_2}$.
  - In particular, if $a_1 = a_2 = a$, the solution is
    $x \equiv a m_2 t + a m_1 s \equiv a(m_2 t + m_1 s) \equiv a \pmod{m_1 m_2}$
- Time complexity is $O(N^3)$, dominated by EEA.

# Examples 2

- Solve $x \equiv 6 \pmod{13}$, $x \equiv 11 \pmod{17}$

| $k$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $r_k$ | 17 | 13 | | | |
| $\begin{pmatrix} 1 \\ -q_{k-1} \end{pmatrix}$ | | | | | |
| $\begin{pmatrix} s_k \\ t_k \end{pmatrix}$ | $\begin{matrix} 1 \\ 0 \end{matrix}$ | $\begin{matrix} 0 \\ 1 \end{matrix}$ | | | |

  - $17 \cdot (-3) + 13 \cdot 4 = 1$
  - $x \equiv 11 \cdot 13 \cdot 4 + 6 \cdot 17 \cdot (-3) \equiv 266 \equiv 45 \pmod{221}$

# Outline

Euclidean Algorithm

Fermat's Little Theorem

# Modular $n$-th Root

- What about modular $n$-th root?

$$x^n \equiv a \pmod{m}.$$

  - Obviously you can solve it via brute-force in $O(2^N)$ time for a $N$-bit $m$. However, this is not what we are interested into.
- Consider the case when $m = p$ is a prime number first.

# Fermat's Little Theorem

- Consider an integer $x$ that is not a multiple of $p$.
- What does the sequence $kx \bmod p$ look like for $k = 1, 2, \ldots, p-1$?
  - A permutation of $1, 2, \ldots, p-1$ since
    - These $p-1$ remainders are all within $1, 2, \ldots, p-1$.
    - They are all different since $p$ is prime.
  - So $x \cdot (2x) \cdots ((p-1)x) \equiv 1 \cdot 2 \cdots (p-1) \pmod{p}$.
- In other words, $(p-1)! x^{p-1} \equiv (p-1)! \pmod{p}$.
  - So $x^{p-1} \equiv 1 \pmod{p}$ since $gcd((p-1)!, p) = 1$.
- Fermat's Little Theorem: $x^p \equiv x \pmod{p}$
  - Also include the case $x \equiv 0 \pmod{p}$.
- Example: $2^{13} \equiv 2 \pmod{13}$, $3^{13} \equiv 3 \pmod{13}$.

# Solve Modular $n$-th Root for Prime $p$

- Solve $x^5 \equiv 2 \pmod{13}$.
  - $x^{10} \equiv 4 \pmod{13}$, $x^{15} \equiv 8 \pmod{13}$, $x^{25} \equiv 6 \pmod{13}$.
  - Fermat's Little Theorem: $x^{13} \equiv x \pmod{13}$
    - So $x^{25} \equiv x^{13}x^{12} \equiv xx^{12} \equiv x \pmod{13}$.
  - Solution: $x \equiv 6 \pmod{13}$
- How about $x^n \equiv a \pmod{p}$?
  - Assume $gcd(n, p-1) = 1$.
    - No, you can't use this method if $n = 2$.
  - Solve $ny \equiv 1 \pmod{p-1}$ for $y$ (via EEA).
  - Solution: $x \equiv a^y \pmod{p}$, or practically $x = a^y \bmod p$.
  - Check: $x^n \equiv a^{ny} \equiv a^{(ny) \bmod (p-1)} \equiv a \pmod{p}$.
- Time complexity
  - EEA takes $O(N^3)$ time.
  - $a^y \bmod p$ can be completed in $O(N^3)$ time. (How?)
  - Overall $O(N^3)$ time again!

# Example

- Solve $x^5 \equiv 10 \pmod{17}$.
- Apply EEA to solve $5y \equiv 1 \pmod{16}$
  - $y \equiv 13 \pmod{16}$
- $x \equiv 10^{13} \pmod{17}$
  - Can we use a calculator to compute $10^{13} \bmod 17$?

# Square-and-Multiply

- Compute $10^{13} \bmod 17$
- $10^{13} \equiv 10^8 \cdot 10^4 \cdot 10^1 \pmod{17}$
  - Since $13 = (1101)_2$
- Use square to calculate $10^2 \bmod 7$, $10^4 \bmod 7$, etc.
  - $10^2 \equiv 100 \equiv 15 \pmod{17}$
  - $10^4 \equiv 225 \equiv 4 \pmod{17}$
  - $10^8 \equiv 16 \pmod{17}$
- So $x \equiv 10^{13} \equiv 16 \cdot 4 \cdot 10 \equiv 11 \pmod{17}$
- Indeed, this algorithm computes $a^y \bmod p$ in $O(N^3)$ time.
  - $O(N)$ modular multiplications.

# Square-and-Multiply by Hand

$$\begin{aligned} 10^{13} &\equiv 10^{12} \cdot 10 \\ &\equiv 100^6 \cdot 10 \equiv 15^6 \cdot 10 \\ &\equiv 225^3 \cdot 10 \equiv 4^3 \cdot 10 \\ &\equiv 4^2 \cdot 40 \equiv 4^2 \cdot 6 \\ &\equiv 16 \cdot 6 \equiv 96 \equiv 11 \pmod{17} \end{aligned}$$

- Be creative with your calculators!

# Summary

- EEA is essential for solving modular algebra equations.
  - In particular, if $gcd(a, b) = 1$, we can apply EEA to find integers $s$ and $t$ such that $as + bt = 1$.
- EEA is efficient with a time complexity of $O(N^3)$ for $N$-bit inputs.
- With Fermat's Little Theorem, we are able to solve modular $n$-th root for prime numbers in many cases for $O(N^3)$ time as well.