1. The decision rule is

\[
d(x) = \begin{cases} 
  d_0, & \Lambda(x) < 1 \\
  d_1, & \Lambda(x) > 1
\end{cases}
\]

with

\[
\Lambda(x) = \frac{e^{-|x-A|}}{e^{-|x|}}
\]

\[= e^{|x|-|x-A|}\]

\[= \begin{cases} 
  e^A, & x > A \\
  e^{2x-A}, & 0 < x < A \\
  e^{-A}, & x < 0
\end{cases}\]

Therefore we use

\[d(x) = \begin{cases} 
  d_0, & x > A/2 \\
  d_1, & x < A/2
\end{cases}\]

The detection probability is

\[
P_D = \int_{A/2}^{\infty} \frac{1}{2} e^{-|x-A|} dx
\]

\[= \int_{-A/2}^{\infty} \frac{1}{2} e^{-|x|} dx
\]

\[= \frac{1}{2} + \int_{0}^{A/2} \frac{1}{2} e^{-x} dx
\]

\[= \frac{1}{2} + \frac{1}{2} (1 - e^{-A/2})
\]

\[= 1 - \frac{1}{2} e^{-A/2}.
\]

Thus,

\[P_D = 1 - \frac{1}{2} e^{-A/2}.
\]

The false alarm rate is

\[
P_{FA} = \int_{-A/2}^{A/2} \frac{1}{2} e^{-|x|} dx
\]

\[= \frac{1}{2} e^{-A/2}.
\]

Thus,

\[P_{FA} = \frac{1}{2} e^{-A/2}.
\]
2. As in Problem 1 we have

\[
\Lambda(x) = \frac{e^{-|x-A|}}{e^{-|x|}} = e^{|x|-|x-A|}
\]

\[
= \begin{cases} 
  e^A, & x > A \\
  e^{2x-A}, & 0 < x < A \\
  e^{-A}, & x < 0
\end{cases}
\]

but now the decision rule is

\[
d(x) = \begin{cases} 
  d_0, & \Lambda(x) < \gamma \\
  d_1, & \Lambda(x) > \gamma
\end{cases}
\]

with \(\gamma\) to be determined to make \(P_{FA} = 0.1\). Since \(\Lambda(x)\) increases monotonically with \(x\), we can see that no matter what the value of \(\gamma\), the decision rule will take the form

\[
d(x) = \begin{cases} 
  d_0, & x < x_0 \\
  d_1, & x > x_0
\end{cases}
\]

Noting that

\[
P_{FA} = \int_{x_0}^{\infty} \frac{1}{2} e^{-|x|} dx
\]

\[
= \begin{cases} 
  \frac{1}{2} e^{-x_0}, & x_0 > 0 \\
  1 - \frac{1}{2} e^{-|x_0|}, & x_0 < 0,
\end{cases}
\]

solving for \(x_0\) yields

\[
x_0 = \begin{cases} 
  - \ln (2P_{FA}), & P_{FA} < 1/2 \\
  \ln (2 (1 - P_{FA})), & P_{FA} > 1/2.
\end{cases}
\]

With \(P_{FA} = 0.1\), this yields \(x_0 = 1.61\) so that the NP test is

\[
d(x) = \begin{cases} 
  d_0, & x < 1.61 \\
  d_1, & x > 1.61
\end{cases}
\]

The value of \(P_D\) is

\[
P_D = \int_{x_0}^{\infty} \frac{1}{2} e^{-|x-A|} dx
\]

yielding

\[
P_D = \begin{cases} 
  \frac{1}{2} e^{-(x_0-A)}, & x_0 > A \\
  1 - \frac{1}{2} e^{-|x_0-A|}, & x_0 < A,
\end{cases}
\]

or for our value of \(x_0\)

\[
P_D = \begin{cases} 
  \frac{1}{2} e^{-(1.61-A)}, & A < 1.61 \\
  1 - \frac{1}{2} e^{-[1.61-A]}, & A > 1.61.
\end{cases}
\]
The test does not depend on $A$, because the form of the decision rule (which is independent of $A$) makes the false alarm rate depend only on $p(x|H_0)$, which has no dependence on $A$. Note, however, that $P_D$ does depend on $A$.

3. (a) We have

$$\ln(\Lambda(x)) = \ln\left(\sqrt{\frac{2}{\pi}}\right) + \left(-\frac{1}{2}x^2 + |x|\right).$$

The plot of this function is the following.

![Plot of the function](image)

(b) One can see that for any $\gamma > -0.2$ (approximately), the NP test will have the form

Choose $H_1$ if $b_0 < |x| < b_1$

for some values $b_0 < b_1$ in the range from 0 to 2.

If $\gamma < -0.2$ (approximately), the NP test will have the form

Choose $H_1$ if $-b_0 < x < b_0$

with $b_0 > 2$.

4. In class we noted that for $T(x) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)$,

$$T(x) \sim \begin{cases} 
\mathcal{N}(0, \sigma^2/N), & \text{under } H_0 \\
\mathcal{N}(A, \sigma^2/N), & \text{under } H_1
\end{cases}$$

Thus

$$P_{FA} = \Pr(T(x) > \gamma; H_0)$$

$$= \frac{1}{2} \text{erfc} \left(\sqrt{\frac{N}{2\sigma^2}\gamma}\right)$$

$$= \frac{1}{2} \text{erfc} \left(\sqrt{\frac{N}{2}\gamma}\right)$$

(1)
and

\[ P_D = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{N}{2\sigma^2}} (\gamma - A) \right) = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{N}{2}} (\gamma - A) \right). \]  

(2)

For a given \( N \), we can use equation (1) to determine the smallest \( \gamma \) that achieves \( P_{FA} = 0.01 \) via

\[ \gamma = \sqrt{\frac{2}{N}} \text{erfc}^{-1}(0.02). \]

Then, the smallest value for \( A \) that achieves \( P_D \geq 0.9 \) is found from equation (2) as

\[ A = \gamma - \sqrt{\frac{2}{N}} \text{erfc}^{-1}(1.8). \]

Then

\[ NA^2 = 2 \left( \text{erfc}^{-1}(0.02) - \text{erfc}^{-1}(1.8) \right)^2, \]  

(3)

which is a constant independent of our choice of \( N \). We can for instance choose \( N = 1 \) and then set \( A \) according to equation (3).

5. We choose \( \mathcal{H}_1 \) if

\[ p(x|\mathcal{H}_1)p(\mathcal{H}_1) > p(x|\mathcal{H}_0)p(\mathcal{H}_0). \]

If \( p(\mathcal{H}_1) = 1/2 = p(\mathcal{H}_0) \), we choose \( \mathcal{H}_1 \) if

\[
\begin{align*}
\frac{1}{\sqrt{4\pi}} \exp \left( -\frac{1}{4} x^2 \right) &> \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} x^2 \right) \\
\exp \left( -\frac{1}{4} x^2 \right) &> \sqrt{2} \exp \left( -\frac{1}{2} x^2 \right) \\
-\frac{1}{4} x^2 &> \ln(\sqrt{2}) - \frac{1}{2} x^2 \\
\frac{1}{4} x^2 &> \ln(\sqrt{2}) \\
x^2 &> 4 \ln(\sqrt{2}) = \ln(4) \\
|x| &> \sqrt{\ln(4)}.
\end{align*}
\]

Therefore we use

\[
|x| \geq d_1 \geq 1.1774
\]

If \( p(\mathcal{H}_1) = 3/4 \) (and hence \( p(\mathcal{H}_0) = 1/4 \)), then we choose \( \mathcal{H}_1 \) if

\[
\frac{3}{4} \sqrt{\frac{4}{\pi}} \exp \left( -\frac{1}{4} x^2 \right) > \frac{1}{4} \sqrt{\frac{2}{\pi}} \exp \left( -\frac{1}{2} x^2 \right)
\]
and the same type of analysis results in

\[ \exp \left( \frac{1}{4} x^2 \right) > \frac{\sqrt{2}}{3}. \]

Since the right hand side is less than one, this expression holds for all \( x \), and we always choose \( \mathcal{H}_1 \).

Because \( p(x|\mathcal{H}_1) > p(x|\mathcal{H}_0) \) only for larger \( x \), we have the decision rule in the first case. However, in the second case, \( \mathcal{H}_1 \) is three times more likely to occur than \( \mathcal{H}_0 \), resulting in \( 3p(x|\mathcal{H}_1) > p(x|\mathcal{H}_0) \) holding for all \( x \), including small \( x \). This results in the decision rule for the second case.

6. We solve this using Bayes’ Risk, with assigned risks

\[
\begin{align*}
C_{00} &= 0 \\
C_{01} &= 0 \\
C_{10} &= $10M \\
C_{11} &= $10M - $110M \\
&= -$100M.
\end{align*}
\]

Note that \( C_{0i} = 0 \) for both \( i = 0 \) and \( i = 1 \) because we incur no cost and make no profit if we don’t excavate. For \( C_{1i} \), we always incur the excavation cost (\( i = 0 \) and \( i = 1 \)) but reap profits only if mineral deposits are present (\( i = 1 \)).

Since \( C_{10} > C_{00} \) and \( C_{01} > C_{11} \), we have

\[
\frac{p(x|\mathcal{H}_1)}{p(x|\mathcal{H}_0)} \overset{d_1}{\lesssim} \frac{(C_{10} - C_{00})p(\mathcal{H}_0)}{(C_{01} - C_{11})p(\mathcal{H}_1)}
\]

\[
\frac{1}{\sqrt{4\pi}} \exp \left( -\frac{1}{4}(x - 10)^2 \right) \overset{d_1}{\lesssim} \frac{(10)(0.8)}{(100)(0.2)}
\]

\[
\frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2}(x - 8)^2 \right) \overset{d_1}{\lesssim} 0.4.
\]

Noting that

\[
\frac{1}{4} x^2 - 3x + 7 = \frac{1}{4} (x - 6)^2 - 2,
\]

we have that

\[
\frac{1}{4} (x - 6)^2 - 2 \overset{d_1}{\lesssim} \ln \left( 0.4\sqrt{2} \right)
\]

\[
|x - 6| \overset{d_1}{\lesssim} \sqrt{8 + 4 \ln(0.4\sqrt{2})}
\]

yielding
If the mineral deposit value is $20M, then $C_{11} = -10M$, and the decision rule becomes

$$\frac{1}{\sqrt{2}} \exp \left( \frac{1}{4} x^2 - 3x + 7 \right) \begin{cases} \frac{d_1}{d_0} > 4 \\ \frac{d_1}{d_0} \leq 4 \end{cases}$$

or

$$|x - 6| \begin{cases} \frac{d_1}{d_0} > \sqrt{8 + 4\ln(4\sqrt{2})} \\ \frac{d_1}{d_0} \leq \sqrt{8 + 4\ln(4\sqrt{2})} \end{cases}$$

yielding

$$|x - 6| \begin{cases} \frac{d_1}{d_0} > 3.864 \\ \frac{d_1}{d_0} \leq 3.864 \end{cases}$$

Because the expected return on a successful excavation is much less in this case, we want the measurement to indicate a stronger likelihood of $H_1$ in order to justify excavation. Thus, the measurement $x$ must have shifted closer to the mean 10 ($x - 6 > 3.864$), or, because the variance is higher with a deposit, the measurement $x$ must be far enough into the tail of the distribution on the left so that a deposit is sufficiently more likelyn ($6 - x > 3.864$).

7. In this case with equal priors, we use a ML detector. The rule is (by symmetry)

$$d(x(0)) = \begin{cases} d_0, & x < -1/2 \\ d_1, & -1/2 < x < 1/2 \\ d_2, & 1/2 < x. \end{cases}$$
The probability of error is

\[ P_e = P(x > -1/2|\mathcal{H}_0)P(\mathcal{H}_0) + P(x < -1/2|\mathcal{H}_1)P(\mathcal{H}_1) + P(x > 1/2|\mathcal{H}_1)P(\mathcal{H}_1) + P(< 1/2|\mathcal{H}_2)P(\mathcal{H}_2) \]

\[ = \frac{1}{3} \left[ \int_{-1/2}^{\infty} \frac{1}{2} e^{-(x+1)} \, dx + \int_{-\infty}^{-1/2} \frac{1}{2} e^{-|x|} \, dx + \int_{1/2}^{\infty} \frac{1}{2} e^{-|x|} \, dx + \int_{-\infty}^{1/2} \frac{1}{2} e^{-|x-1|} \, dx \right] \]

\[ = \frac{2}{3} \left[ \int_{1/2}^{\infty} \frac{1}{2} e^{-|x|} \, dx + \int_{-\infty}^{-1/2} \frac{1}{2} e^{-|x|} \, dx \right] \]

\[ = \frac{2}{3} \left( 1 - \int_{-1/2}^{1/2} \frac{1}{2} e^{-|x|} \, dx \right) \]

\[ = \frac{2}{3} \left( 1 - \int_{0}^{1/2} e^{-|x|} \, dx \right) \]

\[ = \frac{2}{3} \left( 1 - [1 - e^{-1/2}] \right) \]

\[ = \frac{2}{3} e^{-1/2} \]

and we have

\[ P_e = \frac{2}{3} e^{-1/2} = 0.4044 \]