Final Exam
6 December 2016

Name: _____________________________

This is an open-book, open-notes exam. The use of electronic calculators is permitted. The exam lasts 120 minutes.

The exam consists of four problems worth 25 points each. The total number of points is 100. Do all your work on these pages and indicate your final answers clearly. An extra worksheet is provided for each problem. One extra page is also included at the back, and you may use the backs of the sheets if necessary. (If using the extra page or the backs of pages, be sure to mark clearly what problem that work goes with.)

Neatness and clarity are important and can influence your grade!

Grades

1. _______________ (25 pts.)
2. _______________ (25 pts.)
3. _______________ (25 pts.)
4. _______________ (25 pts.)
Total _______________ (100 pts.)
1. **[25 points]** During your recent adventure travel trip you were bitten by a kind of rat called Sabine’s Rat, and you know that 20% of people that are bitten by these rats contract Thayer’s disease, which is usually fatal if untreated. You have two options. The first is to wait to see if symptoms of Thayer’s disease develop, and if they do, undergo the standard treatment, which, though 100% effective, costs $90,000. Your second option is to undergo a proactive treatment immediately, also 100% effective as long as started prior to the onset of symptoms, for a cost of $2,000. The proactive treatment is 0% effective after symptoms develop. To aid in your decision, there is a test for the disease, which costs $100 for each implementation. Each test gives either a positive result \((x = 1)\), meaning the disease is indicated, or a negative result \((x = 0)\) meaning it is not. With \(H_0\) the null hypothesis of no disease and \(H_1\) the alternate hypothesis of disease, the probabilities for the test result \(x\) are as follows.

\[
P(x|H_0) = \begin{cases} 
0.9, & x = 0 \\
0.1, & x = 1 
\end{cases}
\]

\[
P(x|H_1) = \begin{cases} 
0.3, & x = 0 \\
0.7, & x = 1 
\end{cases}
\]

Given that you undergo \(N\) independent tests, determine a decision rule to apply to the test results that minimizes the expected total cost of testing and treatment (under the constraint that you must undergo the standard treatment if symptoms develop).
EXTRA WORKSHEET for problem 1
2. [25 points] You have a set of IID observations $x(n)$ for $n = 0, \ldots, N - 1$ drawn from a Rayleigh pdf

$$p(x(n); \sigma^2) = \begin{cases} \frac{x(n)}{\sigma^2} \exp \left( -\frac{1}{2} \frac{x^2(n)}{\sigma^2} \right), & x(n) > 0 \\ 0, & x(n) < 0 \end{cases}$$

Find a sufficient statistic for $\sigma^2$ having the smallest possible dimension.
EXTRA WORKSHEET for problem 2
3. **[25 points]** You are given the posterior pdf

\[ p(\theta|x) = \begin{cases} 
\exp[-(\theta - x)], & \theta > x \\
0, & \theta < x 
\end{cases} \]

Find (as a function of \(x\)) the minimum MSE estimator for \(\theta\) and also the MAP estimator for \(\theta\).
EXTRA WORKSHEET for problem 3
4. [25 points] You transmit a radar signal

\[ s(n) = \alpha_0 + \alpha_1 n \]

which strikes a target and returns back to you as a received signal

\[ x(n) = s(n - \Delta) + w(n) \]

where \( w(n) \sim \mathcal{N}(0, \sigma^2) \) IID. Since the signal travels the distance \( D \) between the radar and the target twice during the transit time \( \Delta \), the distance is the velocity \( v_0 \) of the signal times one half \( \Delta \). Propose an estimator for \( D \) based on \( N \) measurements \( x(0), x(1), \ldots, x(N-1) \). The parameters \( \alpha_0, \alpha_1, \sigma^2, \) and \( v_0 \) are known. Is your estimate optimal in any sense?
EXTRA WORKSHEET for problem 4
EXTRA WORKSHEET (indicate problem number clearly)